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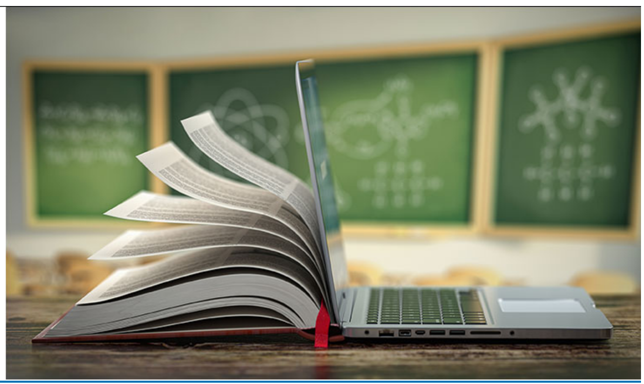

Properties of solutions of dynamic control reconstruction problems

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Abstract. This paper is devoted to inverse problems of the control theory, namely, the dynamic control reconstruction problem. It is the problem of online reconstruction of unknown controls (the input) using known inaccurate measurements of the realized trajectory (the output). Deterministic affine controlled systems are considered. A method for solving this problem is suggested. It relies on auxiliary variational problems on extremum of a regularized integral residual functional. The key feature of this method is using a functional which is convex in control variables and concave in state variables. Properties of the solutions obtained by this method are studied. It is shown that the obtained solutions have oscillating character and are bounded. Results of numerical simulations are provided.

1. Introduction

The dynamic control reconstruction problem (the DCRP) is considered for deterministic affine controlled systems. The admissible controls are bounded measurable functions.

The reconstruction is carried out on the basis of sets of discrete inaccurate measurements of a so-called basic trajectory of the controlled system. This trajectory is being generated by an unknown admissible control. The DCRP consists of online reconstruction of the normal control. The normal control is a control that generates the basic trajectory and has the least possible L^2 norm. An online algorithm of reconstruction that is synchronized with the arrival of measurement points is suggested.

The DCRPs arise in many areas that rely upon the control theory. They include, but are not limited to mechanics [1], aeronautics [2], economics [3, 4] and applied medicine [5].

A well-known approach to DCRPs has been suggested by A.V. Kryazhimskii and Yu. S. Osipov (the KO approach) [6]. There exist several methods for solving these problems based on the KO approach, which are surveyed in [7]. This approach relies upon the so-called extremal aiming procedure. This idea has roots in the works of N. N. Krasovskii's school on the theory of optimal feedback control [8].

Another approach to DCRPs has been suggested and justified [3, 5, 9, 10, 11] by N. N. Subbotina, E. A. Krupennikov and T. B. Tokmantsev (the SKT approach). This approach uses auxiliary variational problems (AVPs) on minimum of an integral residual functional. As well as in the KO approach, a variation of Tikhonov regularization [12] is applied. The developed algorithms [5, 11] based on the SKT approach reduce a DCRP to numerical solving of linear ODEs.

The key feature of the SKT approach is using convex-concave functionals in the AVPs. This paper offers an explanation of the use of such functionals. Namely, it is shown that the solutions of DCRPs obtained by the suggested SKT-algorithms have oscillating character and are bounded, which is provided by the specific functional structure.

2. Dynamics

We consider affine controlled systems of the form

$$\frac{dx(t)}{dt} = G(t,x(t))u(t) + f(t,x(t)), \tag{1}$$
$$x : [0,T] \to \mathbb{R}^n, \quad u : [0,T] \to \mathbb{R}^m, \quad m \geq n, \quad t \in [0,T], \quad T < \infty,$$
$$G : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{m\times n}, \quad f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n,$$

where x is the state variables vector (the output parameter) and u is the vector of the controls (the input parameter).

The set of admissible controls U_{adm} consists of bounded measurable functions:

$$U_{adm} = \{u(\cdot) \in L^1(\mathbb{R}^m, [0,T]) : \ u(t) \in \mathbf{U} \subset \mathbb{R}^m, \quad t \in [0,T]\},$$

where $\mathbf{U}$ is a convex compact set.

3. Input Data

It is supposed that we can observe a so-called basic trajectory $x^* : [0,T] \to \mathbb{R}^n$ of the system (1), which is generated by an unknown admissible control. Namely, discrete inaccurate measurements of the basic trajectory that have error δ arrive with step h_δ:

$$\{y_k^\delta : \ \|y_k^\delta - x^*(t_k)\| \leq \delta, \quad t_k = kh_\delta, \quad k = 0,\ldots,K, \quad K = \lceil T/h_\delta \rceil \in N\}, \tag{2}$$

where the notation $\|\cdot\|$ stands for the Euclidean norm.

We introduce the following assumption:

Assumption. *There exist a constant $\delta_0 > 0$ and a compact $\Psi \subset \mathbb{R}^n$ such that:*

(i) For any measurements' error $\delta \in (0,\delta_0]$ and any step $h_\delta \in (0,T]$

$$\bigcup_{k=0,\ldots,K} B_{2\delta_0}[y_k^\delta] \subset \Psi,$$

where $B_r[x]$ is the closed ball of the radius r with the center in x;

(ii) The elements of $G(t,x)$ and $f(t,x)$ are continuously differentiable with respect to all variables on $[0,T] \times \Psi$;

(iii) The rank of $G(t,x)$ equals n on $[0,T] \times \Psi$.

4. Dynamic Control Reconstruction Problem

Now, we state the DCRP for the dynamics (1) and the input data (2) assuming that Assumptions *(i)–(iii)* hold.

Let U^*_{adm} be the set of admissible controls generating the basic trajectory $x^*(\cdot)$. This set may consist of more than one element. Therefore, the control reconstruction problem is incorrect. To state the correct reconstruction problem we consider the normal control $u^*(\cdot)$, that has the least possible L^2 norm:

$$u^*(\cdot) \in U^*_{adm} : \ \|u^*(\cdot)\|_{L^2} = \min_{u(\cdot)\in U^*_{adm}} \|u(\cdot)\|_{L^2}.$$

It has been proven in [5] that if Assumptions *(i)–(iii)* hold, the normal control $u^*(\cdot)$ exists and is unique in this problem. So, we can state the following DCRP:

The dynamic control reconstruction problem. *For any $\delta \in (0,\delta_0]$, $h_\delta \in (0,h_0]$, $k = 1,\dots,K-1$ and a set of measurements $\{y_j^\delta,\ j = 0,\dots,k\}$ (2) to find a control $u^\delta : [0,t_k] \to \mathbf{U}$ such that at the end instant T of the reconstruction process the function $u^\delta : [0,T] \to \mathbf{U}$ satisfies the following relations:*

(i) $u^\delta(\cdot) \in U^*_{adm}$;

(ii) It generates a trajectory $x^\delta : [0,T] \to \mathbb{R}^n$ of the system (1) with the boundary condition $x^\delta(0) = y_0^\delta$ such that

$$\lim_{\delta\to 0} \|x^\delta(\cdot) - x^*(\cdot)\|_C = 0;$$

(iii) It converges to the normal control:

$$u^\delta(\cdot) \xrightarrow{w^*} u^*(\cdot) \text{ as } \delta \to 0,$$

where the notation $\|\cdot\|_C = \|\cdot\|_{C(\mathbb{R}^n,[0,T])}$ stands for the $C(\mathbb{R}^n,[0,T])$ norm and the notation $\xrightarrow{w^*}$ stands for weak- convergence in the space $\big(C(\mathbb{R}^m,[0,T])\big)^*$.*

5. The algorithm

We consequentially construct the solution on the intervals $[t_k,t_{k+1}]$, $k = 0,\dots,K-1$. On a k-th step (e.g. on the time interval $[t_k,t_{k+1}]$) the following sub-steps are performed:

5.1. Sub-step 1. Interpolation of the measurements.

Discrete measurements (2) are interpolated with a continuously differentiable function $y^\delta(\cdot)$:

$$y^\delta : [0,t_k] \to \mathbb{R}^n : \ y^\delta(t_j) = y_j^\delta, \quad j = 0,\dots,k.$$

On each step $y^\delta(\cdot)$ is extended on the corresponding time interval.

5.2. Sub-step 2. Auxiliary variational problem.

The following auxiliary variational problem (the AVP) is introduced: to find a pair of functions $\{x_k(\cdot),u_k(\cdot)\} \in C^1(\mathbb{R}^n,[t_k,t_{k+1}]) \times C^1(\mathbb{R}^m,[t_k,t_{k+1}])$ such that:

(i) They satisfy the equation (1).

(ii) They satisfy the boundary conditions

$$\begin{aligned} x_0(0) = y_0^\delta, \quad u_0(0) = G^+(0,y_0^\delta)\Big(\frac{dy^\delta(0)}{dt} - f(0,y_0^\delta)\Big),\\ x_k(t_k) = y_k^\delta, \quad u_k(t_k) = u_{k-1}(t_k), \quad k = 1,\dots,K-1, \end{aligned} \tag{3}$$

where $G^+ \stackrel{def}{=} G^T(GG^T)^{-1}$ is the generalized matrix inverse [13]. If Assumption *(iii)* holds, $G^+(0,y_0^\delta)$ exists. The condition imposed on $u_0(0)$ in (3) derives from the condition

$$\frac{dx_0(0)}{dt} = \frac{dy^\delta(0)}{dt}.$$

(iii) They provide a minimum for the functional

$$I_k(x(\cdot),u(\cdot)) = \int_{t_k}^{t_{k+1}} \left[-\frac{\|x(t) - y^\delta(t)\|^2}{2} + \frac{\alpha^2\|u(t)\|^2}{2} \right] dt \to \min, \tag{4}$$

where α is a small regularising [12] parameter.

The necessary optimality conditions for the functions $\{x_k(\cdot), u_k(\cdot)\}$ in the AVP can be obtained in the form of Hamiltonian system [14]:

$$\frac{dx_k(t)}{dt} = -\alpha^{-2} G(t, x_k(t)) G^T(t, x_k(t)) s_k(t) + f(t, x_k(t)), \tag{5}$$

$$\frac{s_{k,i}(t)}{dt} = x_{k,i}(t) - y_i^\delta(t) + \alpha^{-2} \langle s_k^T(t), \frac{\partial G(t, x_k(t))}{\partial x_{k,i}} G^T(t, x_k(t)) s_k(t) \rangle + \langle s_k^T(t), \frac{\partial f(t, x_k(t))}{\partial x_{k,i}} \rangle, \tag{6}$$

$$t \in [t_k, t_{k+1}], \quad i \in 1, \ldots, n$$

with the boundary conditions

$$x_0(0) = y_0^\delta, \quad s_0(0) = -\alpha^2 \Big(G(0, y_0^\delta) G^T(0, y_0^\delta) \Big)^{-1} \left(\frac{dy^\delta(0)}{dt} - f(0, y_0^\delta) \right), \tag{7}$$

$$x_k(t_k) = y_k^\delta, \quad s_k(t_k) = s_{k-1}(t_k), \quad k = 1, \ldots, K-1, \tag{8}$$

where $s_k : [t_k, t_{k+1}] \to \mathbb{R}^n$ is the vector of the adjoint variables.

The function $u_k(\cdot)$ has the form

$$u_k(t) = -\alpha^{-2} G^T(t, x(t)) s_k(t), \quad t \in [t_k, t_{k+1}].$$

5.3. Sub-step 3. Construction of the normal control approximation

We consider a simplified (linearized) version of the Hamiltonian system (5),(6):

$$\frac{dx_k(t)}{dt} = -\alpha^{-2} Q_k s_k(t) + f_k, \tag{9}$$

$$\frac{ds_k(t)}{dt} = x_k(t) - y^\delta(t), \tag{10}$$

$$t \in [t_k, t_{k+1}], \quad Q_k \stackrel{\triangle}{=} G(t_k, y_k^\delta) G^T(t_k, y_k^\delta), \quad f_k \stackrel{\triangle}{=} f(t_k, y_k^\delta) \tag{11}$$

with the boundary conditions (7),(8).

The equations (9),(10) are a linear heterogeneous system of ordinary differential equations with constant coefficients. Therefore, since $y^\delta(t)$ is continuous (by its construction), the boundary problem (9),(10),(7),(8) has a unique solution $\{x_k^{\delta,\alpha}(\cdot), s_k^{\delta,\alpha}(\cdot)\} : [t_k, t_{k+1}] \to \mathbb{R}^n \times \mathbb{R}^n$. This solution is used as a basis for construction of the DCRP solution. Namely, we consider the cut-off function $u^{\delta,\alpha}(\cdot)$:

$$u^{\delta,\alpha}(t) = \begin{cases} \hat{u}^{\delta,\alpha}(t), & \hat{u}^{\delta,\alpha}(t) \in \mathbf{U}, \\ \operatorname*{argmin}_{w \in \mathbf{U}} \|\hat{u}^{\delta,\alpha}(t) - w\|, & \hat{u}^{\delta,\alpha}(t) \notin \mathbf{U}. \end{cases}, \tag{12}$$

$$\hat{u}^{\delta,\alpha}(t) = -\alpha^{-2} G^T(t, y^\delta(t)) s_k^{\delta,\alpha}(t), \ t \in [t_k, t_{k+1}].$$

The following theorem has been proved in [5]:

Theorem. *Let Assumptions* (i)–(iii) *hold. Then the controls $u^{\delta,\alpha}(\cdot)$ of the form (12) solve the DCRP.*

6. Properties of the solution

Let us consider the constructions obtained on a k-th ($k = 0, \ldots, K-1$) step of the algorithm.

We introduce the new variable

$$z_k(t) = x_k(t) - y^\delta(t). \tag{13}$$

In this variable, the system (9),(10) has the form

$$\frac{dz_k(t)}{dt} = -\alpha^{-2}Q_k s_k(t) + f_k - \frac{dy^\delta(t)}{dt}, \tag{14}$$

$$\frac{ds_k(t)}{dt} = z_k(t). \tag{15}$$

If Assumption *(iii)* holds, the rank of the matrix $G(t_k, y_k^\delta)$ equals n. Therefore, Q_k (11) is symmetrical and positive definite [13]. Then, its Schur decomposition has the form $Q_k = R_k\Lambda_k R_k^T$ [13], where the matrix R_k is an orthogonal matrix (that is $R_k^T = R_k^{-1}$) and $\Lambda_k \overset{\triangle}{=} \operatorname{diag}\left(\lambda_1^k, \ldots, \lambda_n^k\right)$ is a diagonal matrix with the positive eigenvalues $\{\lambda_1^k, \ldots, \lambda_n^k\}$ of Q_k on the diagonal.

We introduce another new variables

$$\hat{z}_k(t) = R_k^T z_k(t), \quad \hat{s}_k(t) = R_k^T s_k(t). \tag{16}$$

In these variables, (14),(15) have the form

$$\frac{d\hat{z}_k(t)}{dt} = -\alpha^{-2}\Lambda_k \hat{s}_k(t) + R_k^T\left(f_k - \frac{dy^\delta(t)}{dt}\right), \tag{17}$$

$$\frac{d\hat{s}_k(t)}{dt} = \hat{z}_k(t). \tag{18}$$

Let us consider the matrix of this system M_k:

$$M_k = \begin{pmatrix} O_n & -\alpha^{-2}\Lambda_k \\ E_n & O_n \end{pmatrix} \text{ (in block form)},$$

where E_n and O_n are $n\times n$ identity and zero matrices. The $2n$ eigenvalues of M_k are imaginary numbers $\{i\alpha^{-1}\sqrt{\lambda_1^k}, \ldots, i\alpha^{-1}\sqrt{\lambda_n^k}, -i\alpha^{-1}\sqrt{\lambda_1^k}, \ldots, -i\alpha^{-1}\sqrt{\lambda_n^k}\}$. Therefore, the solution of the homogenous part of the system (17),(18) is

$$\hat{z}_k(t, A_k, B_k) = -Q_k^{\sin}(t)A_k + Q_k^{\cos}(t)B_k, \tag{19}$$

$$\hat{s}_k(t, A_k, B_k) = \alpha\Lambda_k^{\left(-\frac{1}{2}\right)}Q_k^{\cos}(t)A_k + \alpha\Lambda_k^{\left(-\frac{1}{2}\right)}Q_k^{\sin}(t)B_k, \tag{20}$$

$$Q_k^{\sin}(t) \overset{\triangle}{=} \operatorname{diag}\left(\sin\left(\alpha^{-1}\sqrt{\lambda_i^k}(t-t_k)\right), i = 1, \ldots, n\right),$$

$$Q_k^{\cos}(T) \overset{\triangle}{=} \operatorname{diag}\left(\cos\left(\alpha^{-1}\sqrt{\lambda_i^k}(t-t_k)\right), i = 1, \ldots, n\right),$$

$$\Lambda_k^{(p)} \overset{\triangle}{=} \operatorname{diag}\left(\left(\lambda_i^k\right)^p, i = 1, \ldots, n\right) \quad \forall p \in \mathbb{R},$$

where $A_k \in \mathbb{R}^n$, $B_k \in \mathbb{R}^n$ are coefficients, defined by boundary conditions. The expressions (19),(20) show that the solutions of (9),(10) have oscillating character. This property is inherited by the DCRP solution $u^{\delta,\alpha}(\cdot)$ (12).

Let us now consider a particular case, when the interpolation function $y^\delta(\cdot)$ is constructed on each step as a 3-rd order polynomial. In other words,

$$y^\delta(t) = a_k t^3 + b_k t^2 + c_k t + d_k, \quad t \in [t_k, t_{k+1}], \quad (a_k, b_k, c_k, d_k) \in \mathbb{R}^4, \quad k = 0, \ldots, K-1. \tag{21}$$

The following propositions is true.

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Proposition. *If the interpolation function has the form (21), Assumptions* (i)–(iii) *hold and the parameters α and $h_\delta \le h_0$ are chosen in such a way that*

$$\lim_{\alpha\to 0, h_\delta \to 0} \frac{\alpha}{h_\delta} = K_0 \in (0,\infty), \tag{22}$$

then there exist such parameters $R_z = R_z(\alpha,\delta,h_\delta)$, $R_s = R_s(\alpha,\delta,h_\delta)$ such that for any $k = 0,\dots,K-1$ the following estimates are true:

$$\lim_{\alpha\to 0,\delta\to 0,h_\delta\to 0} R_z(\alpha,\delta,h_\delta) < \infty,$$
$$\lim_{\alpha\to 0,\delta\to 0,h_\delta\to 0} R_s(\alpha,\delta,h_\delta) < \infty,$$
$$\|x_k(\cdot) - y^\delta(\cdot)\|_C \le \alpha R_z,$$
$$\|s_k(\cdot)\|_C \le \alpha^2 R_s,$$

where $x_k(\cdot)$, $s_k(\cdot)$ are a solution of (9),(10).

Proof. In the case (21), the derivatives of $y^\delta(t)$ of the order, higher than three, equal zero on the interval $[0,T]$. Therefore, the general form of a solution of (14),(15) is

$$z_k(t) = R_k\hat{z}_k(t,A_k,B_k) - \alpha^2 (Q_k)^{-1} \frac{d^2y^\delta(t)}{dt^2}, \tag{23}$$
$$s_k(t) = -\alpha^2 (Q_k)^{-1}\left(\frac{dy^\delta(t)}{dt} - f_k\right) + R_k\hat{s}_k(t,A_k,B_k) + \alpha^4 (Q_k)^{-1}(Q_k)^{-1}\frac{d^3y^\delta(t)}{dt^3}, \tag{24}$$
$$k = 1,\dots,K-1. \tag{25}$$

Let us consider the first step (k=0). We substitute the boundary conditions (7) to find the coefficients A_0, B_0:

$$z_0(0) = 0 = R_0B_0 - \alpha^2 (Q_0)^{-1}\frac{d^2y^\delta(0)}{dt^2}, \tag{26}$$
$$\begin{aligned} s_0(0) &= -\alpha^2 (Q_0)^{-1}\left(\frac{dy^\delta(0)}{dt} - f_0\right) \\ &= -\alpha^2 (Q_0)^{-1}\left(\frac{dy^\delta(0)}{dt} - f_0\right) + R_0\alpha\Lambda_0^{\left(-\frac{1}{2}\right)}A_0 + \alpha^4 (Q_0)^{-1}(Q_0)^{-1}\frac{d^3y^\delta(0)}{dt^3}. \end{aligned} \tag{27}$$
$$\Rightarrow$$
$$B_0 = \alpha^2\Lambda_0^{(-1)}R_0^T\frac{d^2y^\delta(0)}{dt^2}, \quad A_0 = -\alpha^3\Lambda_0^{\left(-\frac{3}{2}\right)}R_0^T\frac{d^3y^\delta(0)}{dt^3}. \tag{28}$$

We will prove the proposition by induction.

The base of induction: We consider the residuals

$$w_k(t) = s_k(t) - \alpha^2 (Q_k)^{-1}\left(\frac{dy^\delta(t)}{dt} - f_k\right). \tag{29}$$

It follows from (19),(20),(27),(28) that on the first step

$$\|z_0(t)\| = \left\| \alpha^3 R_0 Q_0^{\sin}(t)\Lambda_0^{\left(-\frac{3}{2}\right)} \frac{d^3y^\delta(0)}{dt^3} + \alpha^2 R_0 Q_0^{\cos}(t)\Lambda_0^{(-1)} R_0^T \frac{d^2y^\delta(0)}{dt^2} - \alpha^2 R_0 \left(Q_0\right)^{-1} \frac{d^2y^\delta(0)}{dt^2} \right\| \le \alpha^2 R_0,$$

$$\|w_0(t)\| = \left\| -\alpha^4 R_0 \Lambda_0^{(-2)} Q_0^{\cos}(t) R_0^T \frac{d^3y^\delta(0)}{dt^3} + \alpha^3 R_0 \Lambda_0^{\left(-\frac{3}{2}\right)} Q_0^{\sin}(t) R_0^T \frac{d^2y^\delta(0)}{dt^2} \right\| \le \alpha^3 R_1,$$

$$R_0 \stackrel{\triangle}{=} \alpha R_Q^{\left(-\frac{3}{2}\right)} Y_\delta^{(3)} + 2R_Q^{(-1)} Y_\delta^{(2)},$$

$$R_1 \stackrel{\triangle}{=} \alpha R_Q^{(-2)} Y_\delta^{(3)} + R_Q^{\left(-\frac{3}{2}\right)} Y_\delta^{(2)},$$

$$R_Q^{(-p)} \stackrel{\triangle}{=} \max_{k=0,\dots,K-1} \left(\| \left(Q_k\right)^{-1} \|_2 \right)^{-p},\ p \in (0,\infty), \quad Y_\delta^{(r)} = \max_{k=0,\dots,K-1} \max_{t\in(t_k,t_{k+1})} \left\| \frac{d^r y^\delta(t)}{dt^r} \right\|,\ r \in \mathbb{N},$$

where the notation $\|\cdot\|_2$ stands for the spectral matrix norm. Note that $\|R_k\|_2 = \|R_k^T\|_2 = 1$, since it is an orthonormal matrix [13], $\|Q_k^{\cos}(t)\|_2 \le 1,\ \|Q_k^{\sin}(t)\|_2 \le 1,\ t \in [0,T],\ k = 1,\dots,K-1$.

The step of induction: Let us repeat the calculations (26)-(28) for the k-th step:

$$\begin{aligned}
z_k(t_k) &= 0 = R_k B_k - \alpha^2 \left(Q_k\right)^{-1} \frac{d^2y^\delta(t_k)}{dt^2}, \\
s_k(t_k) &= -\alpha^2 \left(Q_k\right)^{-1} \left(\frac{dy^\delta(t_k)}{dt} - f_k \right) + R_k \alpha \Lambda_k^{\left(-\frac{1}{2}\right)} A_k + \alpha^4 \left(Q_k\right)^{-1} \left(Q_k\right)^{-1} \frac{d^3y^\delta(t_k)}{dt^3} \\
&= s_{k-1}(t_k) = -\alpha^2 \left(Q_{k-1}\right)^{-1} \left(\frac{dy^\delta(t_k)}{dt} - f_{k-1} \right) + w_{k-1}(t_k) \\
&= \alpha^2 \underbrace{\left[\left(Q_k\right)^{-1} \left(\frac{dy^\delta(t_k)}{dt} - f_k \right) - \left(Q_{k-1}\right)^{-1} \left(\frac{dy^\delta(t_k)}{dt} - f_{k-1} \right) \right]}_{\triangle Q_k} \\
&\quad - \alpha^2 \left(Q_k\right)^{-1} \left(\frac{dy^\delta(t_k)}{dt} - f_k \right) + w_{k-1}(t_k)
\end{aligned}$$

$$\Rightarrow$$

$$B_k = \alpha^2 \Lambda_k^{(-1)} R_k^T \frac{d^2y^\delta(t_k)}{dt^2}, \quad A_k = -\alpha^3 \Lambda_k^{\left(-\frac{3}{2}\right)} R_k^T \frac{d^3y^\delta(t_k)}{dt^3} - \alpha \Lambda_k^{\left(\frac{1}{2}\right)} R_k^T \triangle Q_k + \alpha^{-1} \Lambda_k^{\left(\frac{1}{2}\right)} R_k^T w_{k-1}(t_k). \tag{30}$$

The elements of $\left(Q(t)\right)^{-1} \left(\frac{dy^\delta(t)}{dt} - f(t) \right)$ are continuously differentiable in each variable if Assumption *(ii)* holds. Therefore, there exists a constant $R_Q > 0$ such that $\|\triangle Q_k\|_2 \le h R_Q,\ k = 0,\dots,K-1$.

We substitute (30) into (23),(24) and obtain that

$$\|z_k(t)\| = \Big\|\alpha^3 R_k Q_k^{\sin}(t)\Lambda_k^{(-\frac{3}{2})}\frac{d^3y^\delta(t_k)}{dt^3} + \alpha^2 R_k Q_k^{\cos}(t)\Lambda_k^{(-1)}R_k^T\frac{d^2y^\delta(t_k)}{dt^2} - \alpha^2 R_k\left(Q_k\right)^{-1}\frac{d^2y^\delta(t_k)}{dt^2}$$
$$+ R_k Q_k^{\sin}(t)\Lambda_k^{(-\frac{1}{2})}R_k^T(-\alpha\triangle Q_k + \alpha^{-1}w_{k-1}(t))\Big\| \le \alpha^{-1}\|w_{k-1}(\cdot)\|_C + \alpha^2 R_0 + \alpha h_\delta R_Q,$$
$$\|w_k(t)\| = \Big\| -\alpha^4 R_k\Lambda_k^{(-2)}Q_k^{\cos}(t)R_k^T\frac{d^3y^\delta(t_k)}{dt^3} + \alpha^3 R_k\Lambda_k^{(-\frac{3}{2})}Q_k^{\sin}(t)R_k^T\frac{d^2y^\delta(t_k)}{dt^2}$$
$$+ R_k Q_k^{\cos}(t)R_k^T(-\alpha^2\triangle Q_k + w_{k-1}(t))\Big\| \le \|w_{k-1}(\cdot)\|_C + \alpha^3 R_1 + \alpha^2 h_\delta R_Q.$$

This means that on the second step

$$\|z_1(\cdot)\|_C \le \alpha^{-1}\|w_0(\cdot)\|_C + \alpha^2 R_0 + \alpha h_\delta R_Q,$$
$$\|w_1(\cdot)\|_C \le \|w_0(\cdot)\|_C + \alpha^3 R_1 + \alpha^2 h_\delta R_Q.$$

On the third step

$$\|z_2(\cdot)\|_C \le \alpha^{-1}\|w_1(\cdot)\|_C + \alpha^2 R_0 + \alpha h_\delta R_Q \le \alpha^{-1}\|w_0(\cdot)\|_C + \alpha^2 R_1 + \alpha h_\delta R_Q + \alpha^2 R_0 + \alpha h_\delta R_Q,$$
$$\|w_2(\cdot)\|_C \le \|w_1(\cdot)\|_C + \alpha^3 R_1 + \alpha^2 h_\delta R_Q \le \|w_0(\cdot)\|_C + 2\alpha^3 R_1 + 2\alpha^2 h_\delta R_Q.$$

On the k-th step

$$\begin{aligned}\|z_k(\cdot)\|_C &\le \alpha^{-1}\|w_0(\cdot)\|_C + (k-1)\alpha^2 R_1 + (k-1)\alpha h_\delta R_Q + \alpha^2 R_0 + \alpha h_\delta R_Q\\ &\le \alpha^2 R_1 + T h_\delta^{-1}\alpha^2 R_1 + \alpha T R_Q + \alpha^2 R_0 + \alpha h_\delta R_Q,\\ \|w_k(\cdot)\|_C &\le \|w_0(t)(\cdot)\|_C + k\alpha^3 R_1 + k\alpha^2 h_\delta R_Q\\ &\le \alpha^3 R_1 + T h_\delta^{-1}\alpha^3 R_1 + \alpha^2 T R_Q.\end{aligned}$$

It follows from the condition (22) that there exist constants $\hat{h} \in (0, h_0]$ and $\hat{\alpha} > 0$ such that for any $h_\delta \le \hat{h}$ and $\alpha \le \hat{\alpha}$ $\alpha h_\delta^{-1} \le 2K_0$ and

$$\|z_k(\cdot)\|_C \le \alpha\left(\alpha R_1 + 2TK_0R_1 + TR_Q + \alpha R_0 + h_\delta R_Q\right) \stackrel{\triangle}{=} \alpha R_z,$$
$$\|w_k(\cdot)\|_C \le \alpha^2\left(\alpha R_1 + 2K_0TR_1 + TR_Q\right) \stackrel{\triangle}{=} \alpha^2 R_w,$$
$$k = 0,\dots,K-1$$

We apply this estimate to the residual (29) and finally obtain that

$$\|s_k(\cdot)\|_C \le \alpha^2\left(R_w + R_Q^{(-1)}(Y_\delta^{(1)} + R_f)\right) \stackrel{\triangle}{=} \alpha^2 R_s, \quad k = 1,\dots,K-1,$$
$$R_f \stackrel{\triangle}{=} \max_{\{t,x\}\in[0,T]\times\Psi}\|f(t,x)\|.$$

The proposition is proven. □

The next corollary follows directly from the proposition and (12).

Corollary. *The controls $u^{\delta,\alpha}(\cdot)$, defined in (12), satisfy the following estimates:*

$$\|u^{\delta,\alpha}(\cdot)\|_{C(\mathbb{R}^m,[0,T])} \le R_u \stackrel{\triangle}{=} R_s \max_{\{t,x\}\in[0,T]\times\Psi}\|G(t,x)\|_2 < \infty.$$

The expressions (20),(24),(12) show that the solution has oscillating character, which is provided by the form of the functional (4).

To better illustrate that the sign of the residual in the functional affects the character of the solutions, let us consider an AVP with a convex functional:

$$I_k^+(x(\cdot),u(\cdot)) = \int_{t_k}^{t_{k+1}} \left[\frac{\|x(t)-y^\delta(t)\|^2}{2} + \frac{\alpha^2\|u(t)\|^2}{2} \right] dt \to min.$$

The linearized Hamiltonian system (9),(10) in this case will have the form

$$\frac{dx_k(t)}{dt} = -\alpha^{-2}Q_k s_k(t) + f_k, \tag{31}$$

$$\frac{ds_k(t)}{dt} = -(x_k(t) - y^\delta(t)). \tag{32}$$

After we introduce the new variables (13),(16), the matrix of this system will be

$$M_k^+ = \begin{pmatrix} O_n & -\alpha^{-2}\Lambda_k \\ -E_n & O_n \end{pmatrix} \text{ (in block form).}$$

The $2n$ eigenvalues of M_k^+ are real numbers $\{\alpha^{-1}\sqrt{\lambda_1^k}, \ldots, \alpha^{-1}\sqrt{\lambda_n^k}, -\alpha^{-1}\sqrt{\lambda_1^k}, \ldots, -\alpha^{-1}\sqrt{\lambda_n^k}\}$. Therefore, the solutions of (31),(32) will have exponential character, which may result in unstable numerical integration.

7. Example

We consider a simplified model of a flat two-unit manipulator [15]. The dynamics are

$$\ddot{q}(t) = G(t,q(t))u(t), \quad q(t) \in \mathbb{R}^2, \quad u(t) \in \mathbb{R}^2, \tag{33}$$

$$G(t,q) = \begin{pmatrix} I_1 + I_2 + m_1(l_1')^2 + m_2 l_1^2 & m_2 l_1 l_2' \cos(q_1 - q_2) \\ m_2 l_1 l_2' \cos(q_1 - q_2) & m_2(l_2')^2 + I_2 \end{pmatrix},$$

$$|u_1(t)| \le 1, \quad |u_2(t)| \le 1, \quad t \in [0,25],$$

$$l_1 = l_2 = 1m,\ m_1 = 20kg,\ m_2 = 10kg,\ l_1' = 0.5m,\ l_2' = 0.3m,\ I_1 = 1.67kg \cdot m^2,\ I_2 = 1.67kg \cdot m^2.$$

Here $q(t)$ is the state variables vector, representing angle of rotation of the manipulator's arms and $u(t)$ is the controls vector, representing the torque in the joints. For simplicity, the weight of the manipulator is considered to be fully compensated. In other words, the manipulator's arms have zero mass in the model.

The dynamics (33) can be reduced to the form (1) by introducing additional variables $q_3(t) = \frac{dq_1(t)}{dt}$, $q_4(t) = \frac{dq_2(t)}{dt}$.

We state the DCRP for the basic trajectory generated by the controls

$$u_1^*(t) = \begin{cases} t/25, & t \in [0,8], \\ 0.33, & t \in [8,25]. \end{cases}, \quad u_2^*(t) = \begin{cases} 0.2, & t \in [0,8], \\ -0.1, & t \in [8,25]. \end{cases}.$$

The basic trajectory has been constructed numerically and randomly perturbed in discrete points to obtain the inaccurate measurements.

The results of numerical simulations of solving the considered DCRP are presented in Fig.1 for the approximation parameters $\delta = 0.01,\ h_\delta = 0.25,\ \alpha = 0.1$ and in Fig.2 for $\delta = 0.001,\ h_\delta = 0.125,\ \alpha = 0.05$.

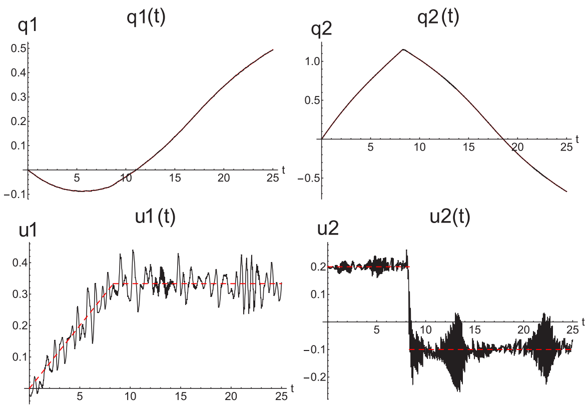

Figure 1. Approximations of the basic trajectory and the normal controls. Dashed lines depict $x^*(t)$ and $u^*(t)$, solid lines depict $x^{\delta,\alpha}(t)$ and $u^{\delta,\alpha}(t)$. $\delta = 0.01,\ h_\delta = 0.25,\ \alpha = 0.1$

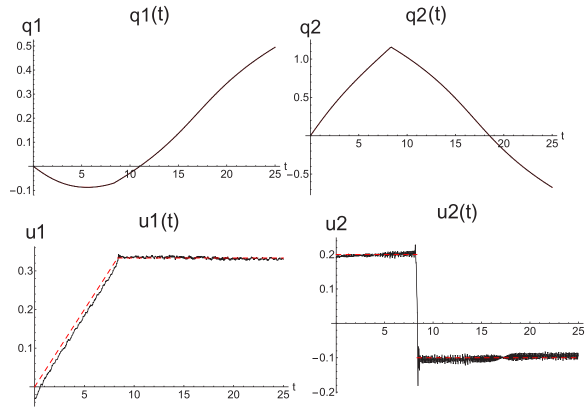

Figure 2. Approximations of the basic trajectory and the normal controls. Dashed lines depict $x^*(t)$ and $u^*(t)$, solid lines depict $x^{\delta,\alpha}(t)$ and $u^{\delta,\alpha}(t)$. $\delta = 0.001,\ h_\delta = 0.125,\ \alpha = 0.05$

8. Conclusion

It has been shown that the suggested solutions of the DCRP have oscillating character and are bounded. This specific structure of the solutions is the reason of the use of a convex-concave functional (4) in the AVP.

Results of numerical simulations are provided on the example of solving the DCRP for a mechanical model.

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References

[1] Subbotina N N and Krupennikov E A 2017 *Proc. Steklov Inst. Mathematics.* **299**, Suppl. 1 205–16
[2] Subbotina N N and Krupennikov E A 2018 *AIP Conf. Proc.* **1997** ed Ashyralyev A, Lukashov A and Sadybekov M (USA: AIP Publishing)
[3] Subbotina N N, Tokmantsev T B and Krupennikov E A 2017 *IFIP Adv. Inf. Comm. Te.* **494** 472–81
[4] Krupennikov E A 2018 *Frontiers of Dynamic Games* (Berlin: Springer) 89–120
[5] Subbotina N N and Krupennikov E A 2020 *Minimax Theory and its Applications* **5** no. 2 439–54
[6] Kryazhimskij A V and Osipov Yu S 1983 *Eng. Cybern.* **21(2)**, 38–47
[7] Osipov Yu S, Kryazhimskii A V and Maksimov V I 2011 *P. Steklov Inst. Math.* **275** 86–120
[8] Krasovskii N N and Subbotin A I 1998 *Game-Theoretical Control Problems* (New York: Springer-Verlag)
[9] Krupennikov E A 2018 *IFAC-PapersOnLine* **51** Iss. 32 343–48.
[10] Krupennikov E A 2019 *Comm. in Comp. and Inf. Sc.: Math. Opt. Th. and Oper. Res.* **1090** ed Khachay M, Kochetov Yu and Pardalos P 508–23.
[11] Subbotina N N 2020 *Stability, Control and Differential Games. Lect. Notes Contr. Inf.* ed Tarasyev A, Vyacheslav M and Filippova T (Berlin: Springer) 367–77
[12] Tikhonov 1943 *A N Dokl. Akad. Nauk SSSR* **39** 195–98
[13] Neudecker H and Magnus J R 2019 *Matrix Differential Calculus with Applications in Statistics and Econometrics* Third ed. (New York: John Wiley & Sons Ltd)
[14] Osipov Yu S, Vasilev F P and Potapov M M 1999 *Basics of the dynamic regularization method* (Moscow: Izd. Moskovskogo universiteta)
[15] Zenkevitch S L and Yushenko A S 2000 *Robot controlling. Basics of manipulator robots control.* (Moscow: MSTU Publ.)